An upper triangular matrix A is invertible if and only if all of its diagonal elements are nonzero.

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Contents

0.1 The Claim

Claim: An upper triangular matrix A is invertible if and only if all of its diagonal elements are nonzero.

Proof: The proof follows in three parts.

0.2 The if part

Claim: If all the diagonal elements of an upper triangular matrix A are nonzero then the matrix A is invertible.

Proof:

If A is invertible, then there exists a matrix X such that

$$\mathrm{XA} = \mathrm{I} = \mathrm{Y}$$

where $\mathrm{X} = \mathrm{A}^{-1}$.

Element y_{ij} of the result matrix $\mathrm{Y} = \mathrm{XA}$ is defined as

$$y_{ij} = \sum_{k=1}^{n} x_{ik} a_{kj}$$

where n is the dimensionality of the $n \times n$ matrix A.

Since $\mathrm{Y} = \mathrm{I}$,

$$y_{ij} = \begin{cases} 0 & \text{if } \ i \neq j \\ 1 & \text{if } \ i = j \end{cases}$$

Thus there exist three cases to satisfy: $i = j$, $i < j$, and $i > j$, which shall be considered in turn.

In the first case, $i = j$

$$y_{ii} = \sum_{k=1}^{n} x_{ik}a_{ki} \overset{!}{=} 1$$

For $k > i$, $a_{ki} = 0$ because A is upper triangular and therefore

$$y_{ii} = \sum_{k=1}^{i} x_{ik}a_{ki} \overset{!}{=} 1$$

By setting x_{ik} to 0 when $k < i$ it follows that

$$y_{ii} = x_{ii}a_{ii} \overset{!}{=} 1$$

which is satisfied by setting x_{ii} to $\frac{1}{a_{ii}}$. Therefore by setting x_{ik} to 0 when $k < i$ and by setting x_{ii} to $\frac{1}{a_{ii}}$ the correct value for y_{ij} when $i = j$ is obtained.

Now consider the second case, namely $i < j$

$$y_{ij} = \sum_{k=1}^{n} x_{ik}a_{kj} \overset{!}{=} 0$$

from the argument for the $i = j$ case, $x_{ik} = 0$ when $k < i$, and when $k > j$, $a_{kj} = 0$ because A is upper triangular, so

$$y_{ij} = \sum_{k=i}^{j} x_{ik}a_{kj} \overset{!}{=} 0$$

this can be rewritten as

$$y_{ij} = \sum_{k=i}^{j-1} x_{ik}a_{kj} + x_{ij}a_{jj} \overset{!}{=} 0$$

and therefore by setting

$$x_{ij} = \frac{-\sum_{k=i}^{j-1} x_{ik}a_{kj}}{a_{jj}}$$

it follows that

$$y_{ij} = \sum_{k=i}^{j-1} x_{ik}a_{kj} - \sum_{k=i}^{j-1} x_{ik}a_{kj} = 0$$

and so for $i < j$, $y_{ij} \overset{!}{=} 0$ has been satisified.

The third case occurs when $i > j$

$$y_{ij} = \sum_{k=1}^{n} x_{ik}a_{kj} \overset{!}{=} 0$$

Now, from the $i = j$ case, when $k < i$, $x_{ik} = 0$, but since in this third case $i > j$, or in other words $j < i$, it follows that when

$$k \leq j < i$$

$x_{ik} = 0$, and so for $k \leq j$, $x_{ik} = 0$. But for $k > j$, $a_{kj} = 0$ because A is upper triangular and therefore

$$y_{ij} = \sum_{k=1}^{n} x_{ik} a_{kj} = 0$$

because for $k \leq j$, $x_{ik} = 0$ and for $k > j$, $a_{kj} = 0$. Which means thta when $i > j$, $y_{ij} \overset{!}{=} 0$ is satisfied.

Thus it has been shown how to set the elements of X to satisfy the identity XA = I, and therefore if all of the diagonal elements of an upper triangular matrix A are nonzero, it follows that A is invertible, being what it was required to prove.

0.3 the only if part

Claim: If an upper triangular matrix A is invertible, then all of its diagonal elements are nonzero.

Proof:

First observe that to say A is invertible is to say that there exists a matrix X such that

$$\mathrm{XA} = \mathrm{Y} = \mathrm{I}$$

The proof that if A is invertible then all its diagonal elements are nonzero, is inductive and proceeds from the following claim.

If a_{pp} is nonzero and for all $j \leq p$ it is true that $x_{ij} = 0$ when $i > j$ then $a_{(p+1)(p+1)}$ is nonzero and for all $j \leq (p+1)$ it is true that $x_{ij} = 0$ when $i > j$.

For the base case a_{11} it is necessary to show that a_{11} is nonzero and that for all $j \leq 1$, $x_{ij} = 0$ when $i > j$. Subsequent proof of the inductive step will establish proof of the original claim, namely that all the diagonal elements of A are nonzero.

Consider the element y_{11} of the result matrix XA

$$y_{11} = \sum_{k=1}^{n} x_{1k} a_{k1} \overset{!}{=} 1$$

Since A is upper triangular, when $k > 1$, $a_{k1} = 0$ therefore

$$y_{11} = a_{11} x_{11} \overset{!}{=} 1$$

Both a_{11} and x_{11} must be nonzero for the result to be equal to 1, therefore a_{11} is nonzero.

Now to show that for $j \leq 1$, $x_{ij} = 0$ when $i > j$. Consider the case for $j < 1$, and assume that the claim is false, this means that there exists an $i > j$, where $j < 1$ such that $x_{ij} \neq 0$, however there exists no such x_{ij} since there is no $j < 1$ and therefore the assumption is false and it is true that for $j < 1$, $x_{ij} = 0$ when $i > j$.

Consider the case when $j = 1$, it must be shown that when $i > j$, $x_{ij} = 0$. This is demonstrated by considering the elements y_{i1} for $i > 1$, in this case $y_{ij} \overset{!}{=} 0$ because $i \neq j$.

$$y_{i1} \sum_{k=1} x_{ik} a_{k1} \overset{!}{=} 0$$

However, as before, when $k > 1$, $a_{k1} = 0$ because A is upper triangular and therefore

$$y_{i1} = x_{i1} a_{11} \overset{!}{=} 0$$

but a_{11} was already shown to be nonzero and hence $x_{i1} \overset{!}{=} 0$. Therefore for $i > j$, $x_{i1} = 0$.

Thus, the base case, namely that $a_{11} \neq 0$ and that for $j \leq 1$, $x_{ij} = 0$ when $i > j$, has been proven.

It is now necessary to demonstrate that the inductive step holds. It is required to show that if $a_{pp} \neq 0$ and for $j \leq p$, $x_{ij} = 0$ when $i > j$, that $a_{(p+1)(p+1)} \neq 0$ and for $j \leq (p+1)$, $x_{ij} = 0$ when $i > j$.

Assuming then that the precondition holds, namely that a_{pp} is nonzero and that for $j \leq p$, $x_{ij} = 0$ when $i > j$, consider the element $y_{(p+1)(p+1)}$ which must be equal to 1

$$y_{(p+1)(p+1)} = \sum_{k=1}^{n} x_{(p+1)k} a_{k(p+1)} \overset{!}{=} 1$$

it was claimed that for $j \leq p$, $x_{ij} = 0$ when $i > j$, therefore in the sum above, for $k \leq p$, $x_{(p+1)k} = 0$ since $(p+1)$ is greater than k in each case, therefore

$$y_{(p+1)(p+1)} = \sum_{k=(p+1)}^{n} x_{(p+1)k} a_{k(p+1)} \overset{!}{=} 1$$

however when $k > (p+1)$, $a_{k(p+1)} = 0$ since A is upper triangular, and therefore

$$y_{(p+1)(p+1)} = x_{(p+1)(p+1)} a_{(p+1)(p+1)} \overset{!}{=} 1$$

both terms must thus be nonzero and thus it has been shown that $a_{(p+1)(p+1)}$ is nonzero.

Now it is necessary to show that for $j \leq (p+1)$, it is true that $x_{ij} = 0$ when $i > j$. For $j \leq p$ it was already established that $x_{ij} = 0$ when $i > j$ by the induction step precondition, therefore it has already been shown that for $j < (p+1)$, $x_{ij} = 0$ when $i > j$ and need only be shown for $j = (p+1)$. Consider the elements $y_{i(p+1)}$ when $i > (p+1)$, it is necessary to show that these elements each equal 0

$$y_{i(p+1)} = \sum_{k=1}^{n} x_{ik} a_{k(p+1)} \overset{!}{=} 0$$

since in each case $i > (p+1)$, for $k \leq p$, $x_{ik} = 0$ and therefore

$$y_{i(p+1)} = \sum_{k=(p+1)}^{n} x_{ik} a_{k(p+1)} \overset{!}{=} 0$$

however when $k > (p+1)$, $a_{k(p+1)} = 0$ because A is upper triangular and therefore

$$y_{i(p+1)} = x_{i(p+1)} a_{(p+1)(p+1)} \overset{!}{=} 0$$

yet $a_{(p+1)(p+1)} \neq 0$ and therefore $x_{i(p+1)} \overset{!}{=} 0$ for $i > (p+1)$ and hence it has been shown that when $i > j$, $x_{i(p+1)} = 0$. Therefore for $j \leq (p+1)$, $x_{ij} = 0$ when $i > j$ and the induction step has been proven.

Therefore when an upper triangular matrix A is invertible, all of its diagonal elements are nonzero, being what it was required to prove.

Note, that this induction also proves that if an upper triangular matrix A is invertible, then the inverse X will also be upper triangular since the induction shows that for $j \leq n$, $x_{ij} = 0$ when $i > j$ (n is the size of the $n \times n$ matrix A).

0.4 Conclusion

It was shown first that if all of the diagonal elements of an upper triangular matrix A are nonzero, then that matrix A is invertible.

It was shown second that if an upper triangular matrix A is invertible, then all of its diagonal elements are nonzero.

Thus what was set out to prove has been established, namely that, an upper triangular matrix A is invertible if and only if all of its diagonal elements are nonzero.

Q.E.D